Math 247A Lecture 1 Notes

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1 Review: The Fourier Transform

1.1 Properties of the Fourier transform

This class is called "Classical Fourier Analysis," but for the past 20 years, it has been taught more like "Modern Harmonic Analysis." Our treatment will be no different.

Definition 1.1. The **Fourier transform** of a function $f \in L^1(\mathbb{R}^d)$ is given by

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x)\,dx.$$

Remark 1.1. By the triangle inequality,

$$\|\widehat{f}\|_{L^\infty} \leq \|f\|_{L^1}.$$

We will prove quantitative results about nice sets of functions and extend these results to more general functions via density arguments. What are our "nice" functions?

Definition 1.2. A C^∞ function $f : \mathbb{R}^d \to \mathbb{C}$ is called a **Schwarz function** if $x^\alpha D^\beta f \in L^\infty$ for all multi-indices $\alpha, \beta \in \mathbb{N}^d$. The vector space of all such functions, $\mathcal{S}(\mathbb{R}^d)$, is called the **Schwarz space**.

This says that all the derivatives of f decay faster than any polynomial. Recall that for a multi-index $\alpha \in \mathbb{N}^d$, we denote

$$|\alpha| = \alpha_1 + \cdots + \alpha_d, \qquad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}, \qquad D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}}.$$

The Schwarz space is a Fréchet space with the topology generated by the countable family of seminorms $\{\varphi_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{N}^d}$ with $\varphi_{\alpha,\beta}(f) = \|x^\alpha D^\beta f\|_{L^\infty}$.

Proposition 1.1 (properties of the Fourier transform)**.** *Fix $f \in \mathcal{S}(\mathbb{R}^d)$.*

1. *If $g(x) = f(x - y)$ with $y \in \mathbb{R}^d$ fixed, then $\widehat{g}(\xi) = e^{-2\pi i y \cdot \xi} \widehat{f}(\xi)$.*

Proof.

$$\widehat{g}(\xi) = \int e^{-2\pi i x\cdot\xi} f(x-y)\,dx = \int e^{-2\pi i(x+y)\cdot\xi} f(x)\,dx = e^{-2\pi i y\cdot\xi}\widehat{f}(\xi). \qquad \square$$

2. *Let $g(x) = e^{2\pi i x\cdot\eta} f(x)$ for $\eta \in \mathbb{R}^d$ fixed. Then $\widehat{g}(\xi) = \widehat{f}(\xi-\eta)$.*

Proof.

$$\widehat{g}(\xi) = \int e^{-2\pi i x(\xi-\eta)} f(x)\,dx = \widehat{f}(\xi-\eta). \qquad \square$$

3. *If $f(x) = f(Tx)$ for $T \in \mathrm{GL}_d(\mathbb{R})$, then $\widehat{f}(\xi) = |\det T|^{-1}\widehat{f}((T^{-1})^\top\xi)$.*

Proof.

$$\begin{aligned}
\widehat{f}(\xi) &= \int e^{-2\pi i x\cdot\xi} f(Tx)\,dx \\
&\overset{y=Tx}{=} |\det T|^{-1}\int e^{-2\pi i T^{-1}y\cdot\xi} f(y)\,dy \\
&= |\det T|^{-1}\int e^{-2\pi i y\cdot(T^{-1})^\top\xi} f(y)\,dy \\
&= |\det T|^{-1}\widehat{f}((T^{-1})^\top\xi).
\end{aligned} \qquad \square$$

4. *If $g = \overline{f}$, then $\widehat{g}(\xi) = \overline{\widehat{f}(-\xi)}$.*
5. *If $g = D^\alpha f$ with $\alpha \in \mathbb{N}^d$, then $\widehat{g}(\xi) = (2\pi i\xi)^\alpha\widehat{f}(\xi)$*

Proof. Using integration by parts,

$$\widehat{g}(\xi) = \int e^{-2\pi i x\cdot\xi} D^\alpha f(x)\,dx = (2\pi i\xi)^\alpha\widehat{f}(\xi). \qquad \square$$

6. *If $g(x) = x^\alpha f(x)$ for $\alpha \in \mathbb{N}^d$, then*

$$\widehat{g}(\xi) = \frac{1}{(-2\pi i)^{|\alpha|}} D^\alpha\widehat{f}(\xi).$$

Proof.

$$\begin{aligned}
\widehat{g}(\xi) &= \int e^{-2\pi i x\cdot\xi} x^\alpha f(x)\,dx \\
&= \frac{1}{(-2\pi i)^{|\alpha|}}\int e^{-2\pi i x\cdot\xi}(-2\pi i x)^\alpha f(x)\,dx \\
&= \frac{1}{(-2\pi i)^{|\alpha|}} D^\alpha\widehat{f}(\xi).
\end{aligned} \qquad \square$$

7. *Let $g = k * f$ for $k \in L^1(\mathbb{R}^d)$. Then $\widehat{g}(\xi) = \widehat{k}(\xi)\widehat{f}(\xi)$.*

Proof.

$$\begin{aligned}\widehat{g}(\xi) &= \int e^{-2\pi i x\cdot\xi}(k*f)(x)\,dx \\ &= \iint e^{-2\pi i x\cdot\xi}k(x-y)f(y)\,dy\,dx \\ &\overset{z=x-y}{=} \iint e^{-2\pi i(z+y)\cdot\xi}k(z)f(y)\,dz\,dy \\ &= \widehat{k}(\xi)\widehat{f}(\xi).\end{aligned}$$

□

Remark 1.2. Properties 1, 2, 3, 4, and 7 extend to $f \in L^1(\mathbb{R}^d)$.

Remark 1.3. Property 3 implies that any rotation and/or reflection symmetry $T \in \mathrm{O}_d(\mathbb{R})$ of f is inherited by $\widehat{f}$. Indeed, if $f(x) = f(Tx)$, then

$$\widehat{f}(\xi) = |\det T|^{-1}\widehat{f}((T^{-1})^\top\xi) = \widehat{f}(T\xi).$$

Exercise 1.1. Show that

1. If $f \in \mathcal{S}(\mathbb{R}^d)$, then $\widehat{f} \in \mathcal{S}(\mathbb{R}^d)$.
2. If $f_n \xrightarrow{\mathcal{S}(\mathbb{R}^d)} f$, then $\widehat{f_n} \xrightarrow{\mathcal{S}(\mathbb{R}^d)} \widehat{f}$.

These follow from properties 5 and 6.

1.2 The Riemann-Lebesgue lemma

Lemma 1.1 (Riemann-Lebesgue)**.** *If $f \in L^1(\mathbb{R}^d)$, then $\widehat{f} \in C_0(\mathbb{R}^d)$ (continuous and vanishing at infinity). In particular, $\widehat{f}$ is uniformly continuous.*

Proof. Let $f_n \in \mathcal{S}(\mathbb{R}^d)$ be such that $f_n \xrightarrow{L^1} f$. By the triangle inequality,

$$\|\widehat{f_n} - \widehat{f}\|_{L^\infty} \le \|f_n - f\|_{L^1} \xrightarrow{n\to\infty} 0.$$

Now $\widehat{f_n} \in \mathcal{S}(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d)$, and $C_0(\mathbb{R}^d)$ is closed in L^∞. So $\widehat{f} \in C_0(\mathbb{R}^d)$. □

1.3 Fourier transform of Gaussians

Lemma 1.2. *Let A be a positive-definite, real-symmetric $d \times d$ matrix. Then*

$$\int e^{-x\cdot Ax}e^{-2\pi i x\cdot\xi}\,dx = (\det A)^{-1/2}\pi^{d/2}e^{-\pi^2\xi\cdot A^{-1}\xi}.$$

Proof. A real-symmetric, positive-definite matrix is diagonalizable, so there exists an orthogonal $O \in \mathrm{O}_d(\mathbb{R})$ such that $A = O^\top D O$ with $D = \mathrm{diag}(\lambda_1, \dots, \lambda_\ell)$ with $\lambda_1, \dots, \lambda_d > 0$. Now

$$x \cdot Ax = x \cdot O^\top DOx = Ox \cdot DOx \overset{y=Ox}{=} y \cdot Dy = \sum \lambda_j y_j^2.$$

We have

$$x \cdot \xi \overset{y=Ox}{=} O^{-1}y \cdot \xi = y \cdot O\xi \overset{\eta=O\xi}{=} y \cdot \eta.$$

So

$$\int e^{-x \cdot Ax} e^{-2\pi i x \cdot \xi} \, dx = \int e^{-\sum(\lambda_j y_j^2 - 2\pi i y_j \eta_j)} \, dy.$$

This is a product of 1-dimensional integrals. Let's look at the 1-dimensional integral

$$\begin{aligned} \int_{\mathbb{R}} e^{-\lambda y^2 - 2\pi i y \eta} \, dy &= \int_{\mathbb{R}} e^{-\lambda (y + \frac{\pi i \eta}{\lambda})^2 - \frac{\pi^2 \eta^2}{\lambda}} \, dy \\ &= \int_{\mathbb{R}} e^{-\lambda y^2} e^{-\pi^2 \eta^2/\lambda} \, dy \\ &= \lambda^{-1/2} \pi^{1/2} e^{-\pi^2 \eta^2/\lambda}. \end{aligned}$$

So we get

$$\begin{aligned} \int e^{-x \cdot Ax} e^{-2\pi i x \cdot \xi} \, dx &= \prod_{j=1}^d (\lambda_j^{-1/2} \pi^{1/2} e^{-\pi^2 \eta_j^2/\lambda_j}) \\ &= (\det A)^{-1/2} \pi^{d/2} e^{-\pi^2 \eta \cdot D^{-1} \eta} \\ &= (\det A)^{-1/2} \pi^{d/2} e^{-\pi^2 \xi \cdot O^\top D^{-1} O \xi} \\ &= (\det A)^{-1/2} \pi^{d/2} e^{-\pi^2 \xi \cdot A^{-1} \xi}. \end{aligned}$$

□

Corollary 1.1. *$e^{-\pi|x|^2}$ is n eigenvector of the Fourier transform with eigenvalue 1.*

Proof.

$$[\mathcal{F}(e^{-\pi|x|^2})](\xi) \overset{A=\pi I}{=} e^{-\pi|\xi|^2}.$$

□

1.4 Fourier inversion

Theorem 1.1 (Fourier inversion)**.** *For $f \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$[(\mathcal{F} \circ \mathcal{F}) f](x) = f(-x),$$

or equivalently,

$$f(x) = \int e^{2\pi i x \cdot \xi} \widehat{f}(\xi) \, d\xi.$$

We can think of this as decomposing f into a linear combination of characters with Fourier coefficients.

Proof. We can't use Fubini like we want to because the integrand is not necessarily absolutely integrable. The (standard) trick is to force a Gaussian in there. For $\varepsilon > 0$, let

$$I_\varepsilon(x) = \int e^{-\pi\varepsilon^2|\xi|^2} e^{2\pi i x\cdot\xi} \widehat{f}(\xi)\, d\xi.$$

Then the dominated convergence theorem tells us that $I_\varepsilon(x) \to \int e^{2\pi i x\cdot\xi} \widehat{f}(\xi)\, d\xi$ as $\varepsilon \to 0$. $\square$

Next time, we will complete the proof.